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# EEL 316 Major Test Semester II 2014-2015

*Answer all questions* (Q.1: 40 marks, Q.2: 20 marks, Q.3: 20 marks) Full Marks: 80

1. Consider coherent reception of a 8-PSK signal over an AWGN channel. The received signal $x(t)$ under hypothesis $H_i$ corresponding to message symbol $m_i$, $i = 1, \ldots, 8$, is given by

$$H_i: \quad x(t) = \sqrt{\frac{2E_s}{T_s}} \cos\left(2\pi f_c t + \frac{i\pi}{4}\right) + w(t), \quad 0 \le t < T_s, \quad f_c \gg \frac{1}{T_s},$$

where $T_s$ is the symbol interval, $f_c$ is the carrier frequency, and $w(t)$ is zero-mean white Gaussian noise with p.s.d. $N_0/2$. The messages $m_1, \ldots, m_8$ occur with equal a priori probabilities.

(a) We use the orthonormal basis $\{\phi_1(t), \phi_2(t)\}$, where

$$\phi_k(t) = A\left[(-1)^{k-1}\cos(2\pi f_c t) - \sin(2\pi f_c t)\right], \quad 0 \le t < T_s, \quad k = 1, 2, \quad A > 0,$$

for the signal space $\{s_1(t), \ldots, s_8(t)\}$, where

$$s_i(t) = \sqrt{\frac{2E_s}{T_s}} \cos\left(2\pi f_c t + \frac{i\pi}{4}\right), \quad 0 \le t < T_s,$$

such that the horizontal axis corresponds to $\phi_1(t)$ and the vertical axis corresponds to $\phi_2(t)$. Find $A$. Find $\underline{s}_i$ for $i = 1, \ldots, 8$ corresponding to this basis. [2+8] 10

Let $\underline{s}_i = [s_{i1} \;\; s_{i2}]^T$, $\underline{x} = [x_1 \;\; x_2]^T$, where $x_k = \int_0^{T_s} x(t)\phi_k(t)dt$, $k = 1, 2$.

(b) Express the ML receiver decision rule in terms of $x_1, x_2, s_{i1}, s_{i2}$ in its simplest form. [4] 3

(c) If the ML receiver is implemented as

$$\hat{i} = \arg\left\{\max_i \; C_i \int_0^{T_s} x(t)\cos(2\pi f_c t)dt + D_i \int_0^{T_s} x(t)\sin(2\pi f_c t)dt\right\},$$

such that $C_i^2 + D_i^2 = 2E_s$, then find $C_i$ and $D_i$ in terms of $s_{i1}, s_{i2}$. [6] 4

(d) Given that $s_1(t)$ is transmitted, find the joint p.d.f. $f(\underline{x} \,|\, \underline{s}_1) = f(x_1, x_2 \,|\, \underline{s}_1)$. [6] 4

Let $q_i = \text{Prob}\,[\underline{x} \in Z_i \,|\, \underline{s}_1]$, $i = 1, \ldots, 8$, where $Z_1, \ldots, Z_8$ are the decision regions.

(e) What is the approximate expression for $q_1$ in terms of $E_s/N_0$ for large SNR? [2] 0

(f) Find the bit error probability (BEP) $P_b$ in terms of $q_1$ and $q_2$ for the bit string to symbol mapping [8] 8

$$(000) \to \underline{s}_1, \quad (001) \to \underline{s}_2, \quad (011) \to \underline{s}_3, \quad (010) \to \underline{s}_4,$$
$$(110) \to \underline{s}_5, \quad (111) \to \underline{s}_6, \quad (101) \to \underline{s}_7, \quad (100) \to \underline{s}_8.$$

(g) If the union bound on the symbol error probability (SEP) $P_e$ is expressed as $\sum_{j=1}^{4} a_j Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\theta_j\right)$, where $\theta_1, \theta_2, \theta_3, \theta_4 \in [0, \pi/2]$ and $\theta_1 < \theta_2 < \theta_3 < \theta_4$, then find $a_j, \theta_j$, $j = 1, 2, 3, 4$. [4] 3

2. Consider a 4-PAM system for which the SEP is given by $\frac{3}{2}Q(\rho)$.

   (a) Find the union bound on the SEP in terms of $\rho$. [8]

   (b) For a 16-QAM system whose SNR is four times the SNR of the 4-PAM system, find the SEP in terms of $\rho$. [6]

   (c) Calculate the SEPs of the 4-PAM and 16-QAM systems when the SNR $E_{av}/N_0$ for the 4-PAM system is 15 dB. [6]

3. (a) The BEP of a coherent orthogonal 4-FSK system operating at SNR $E_s/N_0 = 20$ dB is the same as the SEP of a noncoherent orthogonal 4-FSK system. Using appropriate approximations, calculate the SNR (in dB) at which the noncoherent orthogonal 4-FSK system operates. [8]

   (b) A noncoherent orthogonal BFSK system and a coherent orthogonal 8-FSK system operating at the same $E_s/N_0$ have the same SEP. Calculate the $E_s/N_0$ in dB. [8]

   (c) An $M$-FSK system is to be designed such that (1) its bandwidth efficiency does not fall below $(\sqrt{2}-1)$ bits/s/Hz, and (2) its bandwidth efficiency is always lower than that of the corresponding $M$-PSK system. What are the minimum and maximum values of $M$ that are possible? [4]

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**Some Formulae**

- MAP receiver: $\hat{i} = \arg\left\{\max_i \; -\|\underline{x}-\underline{s}_i\|^2 + N_0 \ln p_i\right\}$
- If $X \sim \mathcal{N}(0,1)$, then $\Pr[X > x] = \int_x^\infty f_X(y)dy = Q(x) = 1 - Q(-x)$
- Use the approximation $Q(x) \approx \frac{1}{x\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, $x \geq 2$, wherever applicable.
- PSK: $P_e \approx 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\left(\frac{\pi}{M}\right)\right)$ for large SNR, $M \geq 4$
- PAM: $P_e = P\left(M, \frac{E_{av}}{N_0}\right) = \frac{2(M-1)}{M}Q\left(\sqrt{\frac{6E_{av}}{(M^2-1)N_0}}\right)$
- QAM, $\log_2 M$ even: $P_e = 1 - \left(1 - P\left(\sqrt{M}, \frac{E_{av}}{2N_0}\right)\right)^2$
- coherent orthogonal FSK: union bound $P_e \leq (M-1)Q\left(\sqrt{\frac{E_s}{N_0}}\right)$ (bound tight for large SNR), $P_b = \frac{M/2}{(M-1)}P_e$
- noncoherent orthogonal FSK: $P_e = \sum_{k=1}^{M-1} \frac{(-1)^{k+1}}{(k+1)} \binom{M-1}{k} e^{-\frac{k}{(k+1)}\frac{E_s}{N_0}}$
- $\rho_{\text{PSK}} = \log_2 M$, $\rho_{\text{PAM}} = 2\log_2 M$, $\rho_{\text{FSK}} = \frac{2\log_2 M}{M}$